Midterm 4 Solutions (by Daniil Kliuev)

Problem 1 *Let G be a simple graph, and let e be an edge of G. Let G/e be the graph we obtain from G by contracting the edge e, and then replacing all the created multiple edges by single edges. Let $G \setminus e$ denote the graph we obtain from G by deleting the edge e. Prove that*

$$p_G(x) = p_{G\setminus e}(x) - p_{G/e}(x),$$

where $p_G(x)$, $p_{G\setminus e}(x)$, and $p_{G/e}(x)$ are the chromatic polynomials of the simple graphs G, $G \setminus e$, and G/e, respectively.

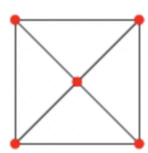
Solution. As two polynomials having an infinite number of common values must be equal, it is enough to prove the statement of the problem for every $x \in \mathbb{N}$. Let $e = uv$. Fix k. Denote by N_1 the number of ways to color $G \setminus e$ with k colors so that u and v have different colors. Denote by N_2 the number of ways to color $G \setminus e$ with k colors so that u and v have the same color. We have $p_{G\setminus e}(k) = N_1 + N_2$.

To color G is to color $G \setminus e$ with additional condition that u and v have different colors. Hence $p_G(k) = N_1$. Suppose that $G \setminus e$ is colored with k colors so that u and v have the same color. Then we define the coloring of G/e as follows: we color all old vertices as in $G \setminus e$ and we color new vertices with the same color as u and v. This gives a proper coloring of G/e.

Suppose that G/e is colored with k colors. Then we define the coloring of $G \setminus e$ as follows. If $w \neq u, v$ we color w as in G/e. We color u and v as new vertex. This gives a proper coloring of $G \setminus e$ such that u and v have the same color. It follows that there are N_2 ways to color G/e with k colors, and so $p_{G/e}(k) = N_2$.

We conclude that $p_G(k) = p_{G\setminus e}(k) - p_{G/e}(k)$. □

Problem 2 *Find the chromatic number and the chromatic polynomial of W_5:*

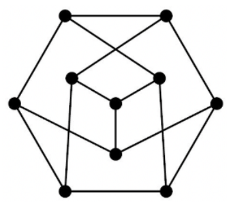


Solution. The graph W_5 contains a triangle, hence its chromatic number is at least 3. Three colors is enough: color the outer cycle using two colors, use the third for central vertex.

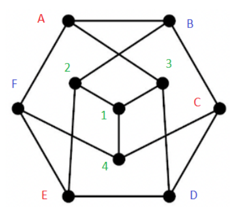
We turn to chromatic polynomial of W_5. Let k be a positive integer. We will find the number of ways to color W_5 using k colors. Let v be the central vertex of W_5. The rest of vertices form a cycle on 4 vertices C_4. First, we choose a color for v; there are

k ways to do this. Since v is connected to all vertices of C_4 this leaves $k-1$ colors for C_4. We know the chromatic polynomial of C_n: it equals to $(x-1)^n + (-1)^n(x-1)$. For $n = 4$ this gives $(x-1)^4 + (x-1)$. Since we use $k-1$ color we replace x with $k-1$ and get $(k-2)^4 + (k-2)$. It follows that $P_{W_5}(k) = k((k-2)^4 + (k-2))$. □

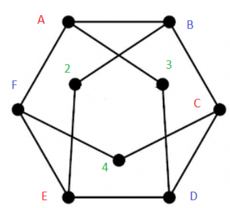
Problem 3 *Is the graph below planar? Justify your answer.*



Solution. Label the vertices as follows:



Delete the central vertex 1. We claim that this subgraph is edge-equivalent to $K_{3,3}$:



We replace vertex 2 and both edges incident to 2 by a new edge connecting B and E. We do the same with vertex 3 and get a new edge connecting A and D. We do the same with vertex 4 and get an edge connecting C and F. We note that these operations preserve edge-equivalence. We obtained a graph that has all edges between blue vertices B, F, D and red vertices A, C, E, in other words $K_{3,3}$. Hence our graph contains a subgraph that is edge-equivalent to $K_{3,3}$. Hence it is not planar. [NOTE: It is not difficult to see that the given graph is actually (isomorphic to) the Petersen graph.] □

Problem 4 *Prove that in every polytope we can always choose two distinct faces having the same number of vertices.*

Solution. Suppose that we have a polytope with e edges and f vertices. Denote faces by $F_1, \dots, F_f$. Let $E(F_i)$ be the number of edges of face F_i, $V(F_i)$ be the number of vertices of face F_i. We have $E(F_i) = V(F_i)$. Suppose that $V(F_1), \dots, V(F_f)$ are distinct. Then we can assume that $V(F_1) < V(F_2) < \dots < V(F_f)$. Each face contains at least three vertices, hence $V(F_1) \geq 3$. It follows that $V(F_i) \geq i + 2$ for all i.

Proposition 34.6 in lecture notes says that $E(F_1) + \ldots + E(F_f) = 2e$. We have

$$E(F_1) + \cdots + E(F_f) = V(F_1) + \cdots + V(F_f) \geq$$
$$3 + 4 + \cdots + (f + 2) = (1 + 2 + \cdots + (f + 2)) - 1 - 2 = \frac{(f + 2)(f + 3)}{2} - 3.$$

In polytopes we have $e \leq 3f - 6$, this is proposition 34.8 in lecture notes. Hence $\frac{(f+2)(f+3)}{2} - 3 \leq 2e \leq 6f - 12$. We deduce that $(f + 2)(f + 3) - 6 \leq 12f - 24$. Rearranging we get $f^2 - 7f + 24 \leq 0$. The discriminant of polynomial $x^2 - 7x + 24$ equals to $7^2 - 4 \cdot 24 < 0$, hence it cannot be nonpositive. We get a contradiction.

□

Problem 5 *Let G be a simple connected graph with $|V(G)| \geq 11$. Prove that either G or the complement of G must be non-planar.*

Solution. Let $|V(G)| = n$. Denote the complement of G by $\overline{G}$. There are $\frac{n(n-1)}{2} - |E(G)|$ in $\overline{G}$. Hence either $E(G)$ or $E(\overline{G})$ is at least $\frac{n(n-1)}{4}$. Changing G to $\overline{G}$ if necessary we can assume that there are at least $\frac{n(n-1)}{4}$ edges in G. Suppose that G is planar. Then $E(G) \leq 3n - 6$. Hence $\frac{n(n-1)}{4} \leq 3n - 6$. We deduce that $n(n - 1) \leq 12n - 24$. From $n \geq 11$ we get $12 \leq n + 1$, hence

$$12n - 24 = 12(n - 2) \leq (n + 1)(n - 2) = n^2 - n - 2 < n^2 - n = n(n - 1).$$

We get a contradiction with $n(n - 1) \leq 12n - 24$.

□